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MULTIMODE BIFURCATIONS OF ELASTIC EQUILIBRIA*

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Conservative elastic systems with parallelepiped symmetry are considered, for which a study of the postcritical equilibria reduces (by the Lyapunov-Schmidt method) to the analysis of extremals of functions of the form

$$W(x_1, \dots, x_n, \lambda) = \Sigma\alpha_j(\lambda) x_j^2 + \Sigma h_{i,j} x_i^2 x_j^2 + \dots$$

where $H = (h_{i,j})$ is a symmetric matrix with non-degenerate principle (diagonal) minors. A relationship is written down for which the matrix H is determined by Ritz approximations of the total energy functional constructed by means of the fundamental bifurcation modes. In the case of soft buckling and for $\operatorname{ind} H = 0$ or $n-1$ (ind is the number of negative eigenvalues taking multiplicity into account) all the allowable types and quantities of bifurcating stable equilibriums are listed. It is shown that after soft buckling with breaking of symmetry, cascade bifurcations are possible (cascade bifurcations simulate the postcritical series of snappings accompanied by a drop in the load /1, 2/). Two known examples of soft buckling and one new example of hard buckling are presented for illustration.

Multimode bifurcations of elastic equilibria were investigated on the basis of a variational (energetic) principle within the framework of problems of the postcritical behaviour of elastic systems /1-3/. The fundamental achievements are obtained here under the influence of the theory of singularities of smooth functions /4, 5/ and ideas associated with the symmetry condition (equivalence of equilibrium equations relative to the action of a group in configuration space) /6-10/. It should be noted that the majority of the results associated with equivalence with respect to a continuous group are obtained by reduction (factorization by means of the group action orbits) to a single-mode bifurcation.

The oft-encountered symmetry of a parallelepiped (equivalence with respect to the action of a group $(Z_2)^n = Z_2 \times \dots \times Z_2$)) results in the analysis of a function that is even in each variable, or equivalently, in the analysis of a function in a cone $R_+^n = \{x \in R^n \mid x_j \geqslant 0\}$ /3, 10, 11/. Up to now, bimodal bifurcations (with the symmetry of a rectangle) reducible to an analysis of functions of the form /5, 12/ $\alpha_1 x_1^2 + \alpha_2 x_2^2 + x_1^4 + a x_1^2 x_2^2 + x_2^4$, $a^2 \neq 4$ have been **investigated** practically completely. In the case of n modes $(n \geqslant 3)$, it has been established for bifurcations reducible to the analysis of functions of the form $(\alpha, y) + (Hy, y) + \dots$, $y = (x_1^2, \dots, x_n^2)^T$ (under the condition of degeneracy of the principal minors of H) that the number of orbits

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of small solutions (taking account of multiplicity and complex solutions) is 2^n. In this situation a study of all the allowable types, of Morse indices, and asymptotic forms of bifurcating solutions as a function of the kind of matrix H is of practical interest. Problems that are traditional in formulations for the general theory of singularities of smooth functions are solved in /9-11/ for functions with group invariance.

1. Formulation of the problem and fundamental results. Let E and F be real Banach spaces, and suppose E is continuously embedded in F. The smooth Fedholm /3, 6/ mapping of f from E into F is called potential if $f = \mathbf{grad_H}\, V$, where $\mathbf{H}$ is a certain Hilbert space in which E and F are embedded compactly and continuously, while V is a smooth functional (mapping potential) on E. If f is included in a smooth parametric family $f(\cdot, \delta)$, $f(x, 0) = f(x)$ of smooth potential mappings with the potential $V(x, \delta)$, $\delta \in R^1$, then the equation

$$f(x, \delta) = 0 \tag{1.1}$$

can be considered as an abstract analogue of the equilibrium equation of a conservative elastic system /13, 14/.

Definition 1.1. Let us say that a buckling condition at zero is satisfied for the Eq. (1.1) for $\delta = 0$ if $f(0, \delta) \equiv 0$, $V(\cdot, \delta)$ takes the value of a strict local minimum at zero for $\delta < 0$ and zero is not the local minimum point for small positive δ. If zero, in addition, is the strict local minimum point for $V(\cdot, 0)$, then we call the buckling at zero soft, otherwise hard.

We assume that $V(\cdot, \delta)$ is continued smoothly to a certain Hilbert (energetic) space $\mathbf{H}^1$, $E \subset \mathbf{H}^1 \subset \mathbf{H}$ (all the embeddings are continuous) with the condition that $\mathbf{grad_{H^1}} V$ is represented in the Leray-Schauder form (one plus a fully continuous mapping from $\mathbf{H}^1$ into $\mathbf{H}^1$). A new stable solution (the point at which $V(\cdot, \delta)$ takes on the minimum locally) appears for such a functional for soft buckling (at zero) in addition to zero. After hard buckling no other minimum point can remain in a previously fixed neighbourhood of zero. This can be seen in the example of a perturbed singularity A_{-3} [5]: $V = -x^4 - \delta x^2$, $x \in R^1$.

In addition let the equation contain the parameters $p \in R^m$:

$$f(x, \delta, p) = 0 \tag{1.2}$$

and let $V(x, \delta, p)$ be the potential for $f(x, \delta, p)$ (the parameter δ takes account of the main load in specific equations, while the parameter p takes account of additional loads, geometric characteristics, etc.). For $f(x, \delta, 0)$ let the buckling conditions at zero for $\delta = 0$ and the following conditions be satisfied:

1.1) For any δ, p the functional $V(\cdot, \delta, p)$ is invariant relative to a set of involutions isometric in $\mathbf{H}$

$$J_k: \mathbf{H} \to \mathbf{H}, \quad J_k(E) \subset E, \quad J_k(F) \subset F, \quad k = 1, \ldots, n$$

1.2) A set of smooth functions normalized in $\mathbf{H}$ (leading bifurcation modes) $\{e_j(\delta, p)\}_{j=1}^n$, $(\delta, p) \in U \subset R^1 \times R^m$ exists for which $J_k(e_k(\delta, p)) = -e_k(\delta, p)$, $J_k(e_j(\delta, p)) = e_j(\delta, p)$, $(\partial f/\partial x)(0, \delta, p)(e_j(\delta, p)) = \alpha_j(\delta, p) e_j(\delta, p)$, $k \neq j$, where $\{\alpha_j(\delta, p)\}_{j=1}^n$ are smooth (spectral) functions;

1.3) The kernel of the operator $(\partial f/\partial x)(0, 0, 0)$ is identical with the linear shell of the vectors $e_1(0, 0), \ldots, e_n(0, 0)$;

1.4) The rank of the matrix comprised of the columns $(\partial\alpha/\partial p_1)(0, 0), \ldots, (\partial\alpha/\partial p_m)(0, 0)$, $(\partial\alpha/\partial\delta)(0, 0)$ where $\alpha(\delta, p) = (\alpha_1(\delta, p), \ldots, \alpha_n(\delta, p))^T$ equals n;

1.5) The inequality $(\partial\alpha_k/\partial\delta)(0, 0) < 0$, $1 \leqslant k \leqslant n$ holds.

Note that $\alpha_1(0, 0) = \ldots = \alpha_n(0, 0) = 0$ follows from 1.3), while the assumption about buckling at zero results from 1.3) and 1.5) if $f(0, \delta, 0) \equiv 0$.

For any $x \in E$ we set $x_j(\delta, p) = \langle x, e_j(\delta, p)\rangle$ (here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{H}$). Then

$$x = \Sigma x_j(\delta, p) e_j(\delta, p) + x^*(\delta, p)$$

Similarly

$$f(x, \delta, p) = \Sigma f_j(x, \delta, p) e_j(\delta, p) + f^*(x, \delta, p)$$

Let $E^*_{\delta, p}$ and $F^*_{\delta, p}$ be orthogonal supplements in E and F in the metric $\langle \cdot, \cdot \rangle$ to the linear shell of vectors $\{e_j(\delta, p)\}_{j=1}^n$. It follows from 1.3) that $f^*(\cdot, 0, 0): E_{0,0}{}^* \to F_{0,0}{}^*$ is a local diffeomorphism. Therefore, by virtue of the theorem on implicit functions a smooth function $x^* = \Phi(\xi, \delta, p)$ is found with values in $E^*_{\delta, p}$, $\xi \in R^n$, for which

$$\Phi(0, 0, 0) = 0, \quad f^*(\Sigma \xi_j e_j(\delta, p) + \Phi(\xi, \delta, p), \delta, p) \equiv 0$$

Definition 1.2. We call the function

$$W(\xi, \delta, p) = V(\Sigma \xi_j e_j(\delta, p) + \Phi(\xi, \delta, p), \delta, p)$$

the key function of Eq.(1.2).

The point $a \in E$ will be the solution of (1.2) for given δ, p if and only if

$a = \Sigma \xi_j e_j(\delta, p) + \Phi(\xi, \delta, p)$, where ξ is the critical point of the key function and $\operatorname{ind}(V(\cdot, \delta, p), a) = \operatorname{ind}(W(\cdot, \delta, p), \xi)$ (ind is the Morse index). Therefore, a study of the solutions of (1.2) in a certain neighbourhood of zero in E for small δ, p reduces to an analysis of the bifurcation of the critical points of the key function $W(\xi, \delta, p)$.

It follows from 1.1) and 1.2) that $W(\xi, \delta, p)$ is even in each variable ξ_j. Therefore

$$W(\xi, \delta, p) = g(\eta, \delta, p), \quad \eta = (\xi_1^2, \ldots, \xi_n^2)^T$$

where $g(\eta, \delta, p)$ is a certain smooth function.

Let us consider the matrix $H(\delta, p)$ which consists of the elements $h_{ij} = \partial^2 g \mid \partial\eta_i \partial\eta_j |_{x=0}$, from which we require satisfaction of the condition

$$\det H_K^\circ \neq 0, \quad K \subset \{1, \ldots, n\} \tag{1.3}$$

Here H_K° is a submatrix consisting of the elements $h_{ij}(0, 0)$, $(i, j) \in K \times K$.

We call $H^\circ = H(0, 0)$ the governing matrix.

If $V(x, \delta, p)$ is represented in the form

$$V(x, \delta, p) = \mathrm{const} + V_{\delta, p}^{(2)}(x) + V_{\delta, p}^{(4)}(x) + o(\|x\|^4) \tag{1.4}$$

where $V_{\delta, p}^{(2)}$ and $V_{\delta, p}^{(4)}$ are homogeneous second- and fourth-order forms then $h_{ij}(\delta, p) = {}^1/_4 \partial^4 W_R(0, \delta, p)/\partial\xi_i^2 \partial\xi_j^2$, where $W_R(\xi, \delta, p)$ is the Ritz approximation of the functional $V(x, \delta, p)$ constructed by means of the system of vectors $\{e_j(\delta, p)\}_{j=1}^n$.

Let the form $V_{\delta, p}^{(4)}$ in the expansion (1.4) be generated by a polylinear symmetric form $V_{\delta, p}^{(4)}(x, y, z, w)$, $V_{\delta, p}^{(4)}(x) = V_{\delta, p}^{(4)}(x, x, x, x)$. Then we obtain for $\overset{\circ}{h}_{i, j}$

$$\overset{\circ}{h}_{i, j} = \begin{cases} 3V_{0,0}^{(4)}(e_j(0, 0), e_j(0, 0), e_i(0, 0), e_i(0, 0)), & i \neq j \\ V_{0,0}^{(4)}(e_j(0, 0)), & i = j \end{cases} \tag{1.5}$$

Formula (1.5) holds for not only potentials of the form (1.4). If the component $V_{\delta, p}^{(3)}$ (the cubic form) is introduced into the expansion (1.4), then representation (1.5) is conserved under the condition

$$\operatorname{grad}_H V_{0,0}^{(3)}(u) = 0, \quad u = \Sigma u_j e_j(0, 0) \tag{1.6}$$

Theorem 1.1 (/3/). Let the matrix H° be positive-definite. Then for sufficiently small δ, p in a sufficiently small neighbourhood O of zero in E there exist not more than 2^n stable solutions of (1.2).

Theorem 1.2. In the case of the positive-definiteness of H° in a certain neighbourhood U of zero of the space of parameters δ, p there is a closed subset σ, nowhere not compact, such that for any point (δ', p') from the fixed connectedness components of the set $U \setminus \sigma$ the number of stable solutions of (1.2) for $\delta = \delta'$ and $p = p'$ is constant and equal to 2^r, $r \in \{0, 1, \ldots, n\}$. For any $r \in \{0, 1, \ldots, n\}$ there is a connectedness component in $U \setminus \sigma$ for which the number of stable solutions equals 2^r.

Theorem 1.3. Let the matrix H° be provisionally positive in a cone R_+^n /15/ and let the index of the quadratic form $(H^\circ x, x)$ equal $n - 1$. Then in a certain neighbourhood U of zero in the space of the parameters δ, p there is a closed subset σ, now here not compact, such that for any point (δ', p') belonging to a fixed connectedness component of the set $U \setminus \sigma$, the number of non-zero stable solutions of (1.2) (in a sufficiently small neighbourhood O of zero in E) is even for $\delta = \delta'$ and $p = p'$ and does not exceed $2n$. For any $r \in \{1, 2, \ldots n\}$ there is a connectedness component in $U \setminus \sigma$ for which the number of stable solutions equals $2r$.

The proof of the theorems will be given in Sect.2.

2. Bifurcation of provisional extremals in a simplicial cone. Investigation of the extremals of the function $W(\xi)$, $\xi \in R^n$, that is even in each variable ξ_j reduces to investigating the conditional extremals in the cone R_+^n for the functions $g(x)$, $x = (\xi_1^2, \ldots, \xi_n^2)^T$, $W(\xi) \equiv g(x)$ /3, 11, 16/.

The point $a \in R_+^n$ is called a conditional critical point (CCP) in R_+^n for the smooth function $g(x)$, $x \in R^n$, if $\operatorname{grad} g(a)$ is orthogonal to the least face of the cone R_+^n containing a. We call the set $\operatorname{supp}(x) = \{j \mid x_j \neq 0\}$, the support of the point $x \in R^n$ and the number $\operatorname{card} \operatorname{supp}(x)$ the order of the point. We denote the subspace $\{x \mid \operatorname{supp}(x) \subset K\}$ by R_K^n, $K \subset \{1, \ldots, n\}$. Therefore $a \in R_+^n$ is the CCP in R_+^n for g if $\operatorname{grad} g(a) \perp R^n_{\operatorname{supp}(a)}$, or, equivalently, the following relationship is satisfied

$$\operatorname{supp}(a) \cap \operatorname{supp}(\operatorname{grad} g(a)) = \varnothing \tag{2.1}$$

If in addition to (2.1) the following equality is satisfied

$$\mathrm{supp}\,(a) \bigcup \mathrm{supp}\,(\mathrm{grad}\, g\,(a)) = \{1, \ldots, n\} \tag{2.2}$$

then we call a the embedded CCP. If the condition of non-degeneracy of the Hessian in a is satisfied for the constraints $g_K = g\,|_{R^n_K}$, $K = \mathrm{supp}\,(a)$, together with (2.1) and (2.2), then we call a a regular point. The CCP that is not regular is called degenerate. A number equal to the usual Morse index of the constraint g_K, $K = \mathrm{supp}\,(a)$ added to the number of negative numbers in the set $\{(\partial g/\partial x_j)\,(a)\}_{j=1}^n$ is called the Morse index of the regular critical point a.

We define the bifurcation set /4, 5/ $\sigma\,(g, O)$, $O \subset R^n$ for the arbitrary smooth evolute $g\,(x, \lambda)$, $g\,(x, 0) = g\,(x)$, $\lambda \in R^m$ as the set λ for which $g\,(\cdot, \lambda)$ has a degenerate CCP in $O \cap R^n_+$. Let $L \subset \{1, \ldots, n\}$, $K \cap L = \varnothing$; $\omega_{K;\,L}\,(g, O)$ be a subset of values of λ for which $g\,(\cdot, \lambda)$ has a regular critical point $a \in O \cap R^n_+$ with support K and for which

$$(\partial g/\partial x_l)\,(a, \lambda) < 0, \quad (\partial g/\partial x_j)\,(a, \lambda) > 0, \quad l \in L, \quad j \notin K \bigcup L$$

It follows from the definition of $\omega_{K;\,L}$ that $\partial\omega_{K;\,L} \subset \sigma\,(g, O)$.

For a function g under the condition (1.3) and the rank of the matrix $(\partial^2 g/\partial\lambda\partial x)\,(0, 0)$ equal to the number n, the following relationship is satisfied

$$\sigma\,(g, O) = \bigcup_{K,\,L} \partial\omega_{K;\,l}\,(g, O)$$

Here and henceforth O is a sufficiently small neighbourhood of zero in R^n.

Moreover, a neighbourhood U of zero is in R^m such that

$$U \cap \sigma\,(g, O) = \bigcup_{K,\,j} (U \cap \sigma_{K;\,j}\,(g, O))$$

where $\sigma_{K,\,j}\,(g, O)$ is a set λ for which $g\,(\cdot, \lambda)$ has a CCP a with support K and

$$j \in \{1, \ldots, n\} \setminus (K \bigcup \mathrm{supp}\,(\mathrm{grad}\, g\,(a, \lambda)))$$

For sufficiently small O and U the existence of not more than one CCP in $O \cap R^n_+$ with the given support follows from (1.3). Furthermore, it is assumed everywhere that

$$\mathrm{grad}\, g\,(0, 0) = 0 \tag{2.3}$$

Lemma 2.1. Let $\mathrm{rank}\,(\partial^2 g/\partial x\partial\lambda)\,(0, 0) = n$ and let conditions (1.3) and (2.3) be satisfied. Then for any neighbourhood O of zero in R^n there is a neighbourhood U of zero in R^m such that for every curve $\lambda\,(t), t \in [0, 1]$, in U intersecting the component $\sigma_{K;\,j}$ for $t = t_1$ and not intersecting any of the remaining components σ for no matter what t, there is a single curve $x\,(t)$ in $O \cap R^n_+$ that consits of a CCP for $g\,(\cdot, \lambda\,(t))$ (for appropriate t) with the constant support K.

By virtue of (1.3) the proof results from the theorem on implicit functions.

We note that the intersection of the components $\sigma_{K;\,j}$ of the curve $\lambda\,(t)$ denotes satisfaction of the equality $(\partial g/\partial x_j)\,(x\,(t_1), \lambda\,(t_1)) = 0$, where $x\,(t)$ is a curve corresponding to $\lambda\,(t)$ according to Lemma 2.1.

Let

$$\gamma\,(t) = \frac{d}{dt}\left(\frac{\partial g}{\partial x_j}\,(x\,(t), \lambda\,(t))\right)$$

Definition 2.1. Let us say that a smooth function $\lambda\,(t)$ intersects $\sigma_{K;\,j}$ positively if $\gamma\,(t_1) > 0$. Otherwise we call the intersection negative.

Lemma 2.2. Let $\mathrm{rank}\,(\partial^2 g/\partial x\partial\lambda)\,(0, 0) = n$ and conditions (1.3) and (2.3) be satisfied. Let the neighbourhoods O and U satisfy the conclusion of the Lemma 2.1 and the smooth curve $\lambda\,(t)$ intersect $\sigma_{K;\,j}$ positively (negatively) for $t = t_1$ and not intersect the remaining components of σ for any t. Then the Morse index of the CCP $x\,(t)$ of the function $g\,(\cdot, \lambda\,(t))$, $\mathrm{supp}\,(x\,(t)) = K$ decreases (increases) by one after the intersection.

Proof. After the positive (negative) intersection, we have $(\partial g/\partial x_j)(x\,(t), \lambda\,(t)) > 0$ (<0) for $t > t_1$. Therefore, the number of negative derivatives decreases (increases) by one. Since the signature of the Hess matrix in $x\,(t)$ is constant in O for the constraint $g_K\,(\cdot, \lambda\,(t))$, then we hence obtain the statement of the lemma.

Lemma 2.3. The CCP branch in R_+^n with support $K \bigcup j$ is either generated from $x\,(t_1)$ or vanishes at this point under the conditions of Lemma 2.2 for the passage of t through t_1. Generation (disappearance) occurs for a positive (negative) intersection and the condition $\det H_K^\circ \det \mathring{H}_{K \cup j} < 0$ or for a negative (positive) intersection and the condition $\det H_K^\circ \det \mathring{H}_{K \cup j} > 0$. The Morse index of the bifurcating point with support $K \bigcup j$ agrees with the Morse index of the point of support K considered prior to the generation (after the disappearance) of the bifurcating branch.

Proof. A critical point with support $K \cup j$ is defined by the following system of equations:

$$(\partial g/\partial x_k)(x, \lambda(t)) = x_q = 0, \quad k \in K \cup j, \quad q \notin K \cup j \tag{2.4}$$

It follows from the conditions of the lemma that this system is solvable for $t = t_1$, and therefore, even for $t \neq t_1$. Let $y(t)$ be a solution of system (2.4) $y(t_1) = x(t_1)$, where $x(t)$ is a CCP branch with support K. Let Γ denote the Hess matrix of the function $g(\cdot, \lambda(t_1))$ at the point $x(t_1)$ and $\Gamma_{K;L}$ its submatrix consisting of $\gamma_{k,l}$, $k \in K$, $l \in L$. For $t = t_1$ we obtain from the rule for differentiation of implicit functions

$$\Gamma_{j;K}\frac{dx_K}{dt} + B_j\frac{d\lambda}{dt} = \gamma, \quad \Gamma_{K;K}\frac{dx_K}{dt} + B_K\frac{d\lambda}{dt} = 0 \tag{2.5}$$

$$\Gamma_{j;K\cup j}\frac{dy_{K\cup j}}{dt} + B_j\frac{d\lambda}{dt} = 0, \quad \Gamma_{K;K\cup j}\frac{dy_{K\cup j}}{dt} + B_K\frac{d\lambda}{dt} = 0 \tag{2.6}$$

$$B_j = \frac{\partial^2 g}{\partial\lambda\partial x_j}, \quad B_K = \left(\frac{\partial}{\partial\lambda}\operatorname{grad} g\right)_K$$

(x_K is a vector comprised of the components x_k, $k \in K$ of the vector x).
We obtain from (2.5) and (2.6)

$$(\Gamma_{j;j} - \Gamma_{j;K}\Gamma_{K;K}^{-1}\Gamma_{K;j})\frac{dy_j}{dt} + \gamma = 0$$

The factor before the derivative in this last expression equals $\det\Gamma_{K\cup j;K\cup j}/\det\Gamma_{K;K}$. Therefore

$$\operatorname{sgn} dy_j/dt = -\operatorname{sgn}\gamma\operatorname{sgn}(\det H_K^\circ \det H_{K\cup j}^\circ)$$

The assertion of the lemma follows from this last equality.

We will now prove Theorem 1.1 and 1.2. If the matrix H° is positive-definite, then it follows from Lemma 2.3 that $g(\cdot, \lambda)$, $\lambda = (\delta, p)$ has a unique zero-index CCP in O. If the curve $\lambda(t)$, $t \in [0, 1]$ connects a point of the domain $\omega_{Z;\varnothing}$ with the point of the domain $\omega_{K;\varnothing}$, then in conformity with Lemma 2.3 a stable point with support K bifurcates during passage into $\omega_{K;\varnothing}$. The set $\omega_{K;\varnothing}$ is given by the relationships

$$\partial g/\partial x_k = x_j = 0, \quad x_k > 0, \quad \partial g/\partial x_j > 0, \quad k \in K; \quad j \notin K$$

and is obviously not empty when conditions (1.3) and (2.3) are satisfied.

If $W(\xi, \lambda) = g(x, \lambda)$, $x = (\xi_1^2, \ldots, \xi_n^2)$, $\lambda = (\delta, p)$, then the critical point $\xi \in R_+^n$, $\xi_j = \sqrt{x_j}$ of the function $W(\cdot, \lambda)$ is in one-to-one correspondence (with the Morse index conserved) with each CCP x in R_+^n of the function $g(\cdot, \lambda)$. By changing the signs in front of the components ξ_j we obtain other (adjacent) critical points of the function $W(\cdot, \lambda)$ with the same Morse index. There will be 2^r of these points, where $r = \operatorname{card}\operatorname{supp}(x)$.

Under the conditions of Theorem 1.3 the CCP of zero index in O is of order not higher than the first. The function $g(\cdot, \nu t)$ has just r stable first order CCP with supports $K = \{k_1, \ldots, k_r\}$, in O for small t if ν is the solution of the equation $(\partial^2 g/\partial x\partial\lambda)(0, 0)\nu = p$, $p = (p_1, \ldots, p_n)^T$, where $p_j = -\sqrt{h_{j,j}}$ for $j \in K$ and $p_j = \sqrt{h_{j,j}}$ for $j \notin K$.

Remark. It follows from the proof of Lemma 2.3 that the component $x_K(\lambda)$ of the critical point $x(\lambda)$ with support K has the following asymptotic representation:

$$x_K(\lambda) = -(H_K^\circ)^{-1} b_K(\lambda) + o(|\lambda|) \tag{2.7}$$
$$b(\lambda) = \operatorname{grad}_{R^n} g(0, \lambda)$$

The relationships

$$x_k(\lambda) > 0, \quad \frac{\partial g}{\partial x_l}(x(\lambda), \lambda) < 0, \quad \frac{\partial g}{\partial x_j}(x(\lambda), \lambda) > 0$$
$$k \in K, \quad l \in L, \quad j \notin K \cup L$$

are the necessary and sufficient conditions for inclusion of the point λ in the domain $\omega_{K;L}$.

3. Remarks about initial imperfections and cascade bifurcations. In a broad sense taking account of initial imperfections means studying possible changes in the nature of the bifurcation during passage from (1.1) to the perturbed equation $f'(x, \delta) = 0$ with the potential $V'(x, \delta)$

$$\| d^j V'(x, \delta) - d^j V(x, \delta) \| < \varepsilon, \quad (x, \delta) \in O \times U, \; j \leqslant m \tag{3.1}$$

Here d^j is the j-th order differential, and m is a given integer.

Let us say that Eq. (1.1) allows an r-step cascade bifurcation (r is a positive integer) if for arbitrarily small neighbourhoods O and U of the zeros in E and R^1 there exists an arbitrarily close perturbed equation $f'(x, \delta) = 0$ in the sense of (3.1) for which a curve $(x(t), \delta(t))$, $t \in [0, 1]$ is in $O \times U$ such that 1) $f'(x(t), \delta(t)) = 0$, 2) the projection $(x, \delta) \to \delta$ contracted on the graph of this curve has just $2r$ turning points (points of local homeomorphism), 3) the Morse index of the potential $V'(\cdot, \delta(t))$ equals zero or one at the point $x(t)$ if

$(x(t),\ \delta(t))$ is not a turning point. Single-stage cascade bifurcations are generated by one-dimensional assemblies /5/.

It is easy to note that an r-stage cascade bifurcation is generated by singular points of the type A_{2r+1} /4, 5/ that appear in specific problems as adjoining to the simplest multi-dimensional singular points (on degenerating into many modes). In this connection the following is of interest

Theorem 3.1. If conditions 1.1)-1.3) are satisfied for $V(g,\delta)$, and the governing matrix H° is positive definite, then a functional $V(x,\delta)$ is as close to $V'(x,\delta)$ as we please (in the sense of (3.1) with any m), for which the key function has the form

$$\sum_{k=1}^{2r} a_k(\delta)\xi^k + o(|\xi|^{2r}),\ \xi\in R^1,\ r=2^n \qquad (3.2)$$
$$\alpha_1(0)=\ldots=\alpha_{2r-1}(0)=0,\quad \alpha_{2r}(0)>0$$

The Morse index of any regular critical point x fairly close to zero for $V'(\cdot,\delta)$ agrees for sufficiently small δ with the Morse index corresponding to its critical point ξ for the function (3.2).

Proof. Let $W(\xi,\delta)$ be the key function for $V(x,\delta)$, $\xi\in R^n$. It follows from the conditions of the theorem that $W(\xi,0)$ is represented in the form $(H^\circ x, x)+o(|x|^2)$, $x=(\xi_1^2,\ldots,\xi_n^2)^T$, where H° is a positive-definite matrix. We consider the function

$$h(\xi,\varepsilon)=g(\xi_1^2,\ \xi_2^2+\varepsilon\xi_1,\ldots,\xi_n^2+\varepsilon\xi_{n-1})$$

and make the change of variables

$$\eta_{k-1}=\varepsilon\xi_{k-1}+\xi_k^2,\quad k=2,\ldots,n;\quad \xi_n=\eta_n$$

The function $h(\cdot,\varepsilon)$ in the variable η_j is semiquasihomogeneous with weights

$$\{{}^1/_2,\ldots,{}^1/_2,\ {}^1/_4q_n^{-1}\}\ [4]\ (q_n=2^{n-1}):$$
$$h(\xi,\varepsilon)=\sum_{i\geqslant 2,\ j\geqslant 2}\mathring{h}_{i,j}\eta_{i-1}\eta_{j-1}+\mathring{h}_{1,1}\varepsilon^{-4(q_n-1)}\eta_n^{4q_n}+ \qquad (3.3)$$
$$2\Sigma\mathring{h}_{1,j}\varepsilon^{-2(q_n-1)}\eta_{j-1}\eta_n^{2q_n}+\omega(\eta_1,\ldots,\eta_n)$$

Here $\omega(\eta_1,\ldots,\eta_n)$ is a combination of power monomials above the Newton polyhedron /4/ of the function $h(\cdot,\varepsilon)$ i.e., *faces containing the exponents of the monomials* $\eta_1^2,\ldots,\eta_{n-1}^2$, $\eta_n^{4q_n}$.

The non-degeneracy of the principal quasihomogeneous part results from the positive-definiteness of H°. And since the corank of the Hess matrix at the zero of the function (3.3) equals one for $\varepsilon\neq 0$, this function has a singular point of the type A_k, $k=4q_n-1=2^{n+1}-1$ at the zero. Hence, the assertion about the form (3.2) follows. The agreement of the Morse indices at corresponding critical points for $V'(\cdot,\delta)$ and (3.2) follows from the positive-definiteness of the principal quasihomogeneous part of the function $h(\cdot,\varepsilon)$.

Definition 3.1. A semihomogeneous fourth-order polynomial of n variables of the form /4/

$$W(\xi)=\sum_{j=1}^{n}\xi_j^4+\sum a_{k_1,\ldots,k_n}\xi_1^{k_1}\circ\ldots\circ\xi_n^{k_n} \qquad (3.4)$$

under the condition that the quartic part of (3.4) is finite-to-one (3^n multiple) is called an assembly of dimensionality n. The summation in the second term in (3.4) is over $k_1,\ldots,k_n$ for $0\leqslant k_j\leqslant 2$, $\Sigma k_j\geqslant 4$.

The introduction of the form (3.4) is motivated by the theory of normal forms of semi-quasihomogeneous functions /4/. A set of polynomials of the form (3.4) forms an affine sub-manifold M in the space of polynomials for coordinates whose points are given by the set of coefficients $a=\{a_{k_1,\ldots,k_n}\}$. The dimensionality of M is $3^n-n(n+1)(n+2)/6-n(n+1)/2-1$.

Let $k(a)$ denote *the greatest of the multiplicities of singularities of the type* A_k adjacent to (3.4) for a given set of coefficients a.

Theorem 3.2. An open, everywhere compact, subset exists in M for any point a for which the following estimate holds

$$k(a)\leqslant n(n+1)/2+n(n+1)(n+2)/6$$

See the proof in /17, 18/.

4. Examples of elastic systems with parallelepiped symmetry. A system of Euler bars. The simplest example of an elastic system with parallelepiped symmetry is a set of identical and identically compressed plane (Eulerian /19, 20/) bars /5/. The governing matrix here is proportional to unity (soft buckling) and therefore, the coexistence of stable modes of equilibrium of any previously assigned but general for all branching modes of the type is allowed in the postcritical phase.

Karman equation for an elastic rectangular plate. Numerical results are presented in /12/ on bimodal bifurcations of solutions of the Karman equation for an axially compressed rectangular plate under different boundary conditions. It follows from this that the matrix H° is conditionally positive in R_+^2 and $\det H^\circ < 0$. In conformity with Theorem 1.3, this means that here the coexistence of stable unimodal (first-order) solutions is allowed in the post-critical stage while the existence of stable bimodal (second-order) solutions is not allowed.

A Kirchhoff rod with elastic reinforcement. We examine a rectilinear axially compressed thin elastic rod in space /19-21/, framed stiffly at the ends and reinforced by an elastic force with the potential

$$\frac{\mu}{2}\left(\int_0^1 (r_2, g_3(s))\,ds\right)^2, \quad r_2 = (0, 1, 0)^T$$

$g_3(s)$ is the direction tangent to the middle line of the rod at the point of the parameter of lengths s, $0 \leqslant s \leqslant 1$, and μ is the parameter of the elasticity force of the reinforcement (reacting to the deviation of the rod end from the axis r_3 in the direction r_2). It is assumed that $\mu > 4\pi^2$. Let $g_1(s)$ and $g_2(s)$ be directions along the principal axes of inertia of a normal section at the point of the middle line of parameter s, and let $r_1 = (1, 0, 0)^T$, $r_3 = (0, 0, 1)^T$.

The Kirchhoff equation /19-21/ of the rod equilibrium configuration with the above-mentioned elastic reinforcement is written in the following form

$$-A\,d\varkappa/ds + [A\varkappa, \varkappa] + \lambda\,[r_3, g^{-1}r_3] + \mu\,\langle r_3, g^{-1}r_2\rangle\,[r_3 g^{-1} r_2] = 0 \tag{4.1}$$

$$\langle \varphi, \psi\rangle = \int_0^1 (\varphi(s), \psi(s))\,ds, \quad A = \operatorname{diag}(A_1, A_2, A_3)$$

Here λ is the parameter of the axial compression force, A is the elasticity tensor in the transverse section for which the E.L. Nikolai condition is satisfied

$$A_3 < \frac{2}{1+\nu}\,\frac{A_1 A_2}{A_1 + A_2}$$

ν is Poisson's ratio $0 < \nu < 1/2$, $\varkappa(s)$ is the angular velocity of section motion as a function of s written in coordinates in the triplet $g_1(s), g_2(s), g_3(s)$; $g(s)$ is a matrix consisting of co-ordinates of the vectors $g_1(s), g_2(s), g_3(s)$ in the basis triplet $g_1(0), g_2(0), g_3(0)$; $[A\varkappa, \varkappa]$ is the vector product.

The boundary condition

$$g(0) = g(1) = I \tag{4.2}$$

corresponds to rigid clamping at the ends.

The potential of (4.1) under condition (4.2) is

$$1/2\langle A\varkappa, \varkappa\rangle + \lambda\,\langle r_3, g^{-1}r_3\rangle + 1/2\mu\,\langle r_3, g^{-1}r_2\rangle^2 \tag{4.3}$$

If $\varkappa$ is identified with the matrix

$$X = \begin{Vmatrix} 0 & -\varkappa_3 & \varkappa_2 \\ \varkappa_3 & 0 & -\varkappa_1 \\ -\varkappa_2 & \varkappa_1 & 0 \end{Vmatrix}$$

then the equality $X(s) = g^{-1}(s)(dg/ds)(s)$ is true for the matrix image $X(s)$ of the vector $\varkappa(s)$ /22/.

Let the functional $V(\varphi, \lambda, \mu, A)$ be obtained from (4.3) by the substitution

$$g = \exp(\varphi_3 r_3)\exp(\varphi_2 r_2)\exp(\varphi_1 r_1), \quad \exp(\varkappa) = \Sigma\,\frac{1}{k!}\,X^k$$

We assume E to be the space of functions $\varphi(s)$, $s \in [0, 1]$ of the class C^2 with values in $\mathbf{R}^3$ that satisfy the condition

$$\varphi(0) = \varphi(1) = 0; \quad F = C([0, 1], R^3), \quad \mathbf{H} = L_2([0, 1], R^3)$$

(the space of continuous and the space of square summable functions in $[0, 1]$ with values in R^3). The functional V is invariant under the involutions J_1, J_2 where $J_1(\varphi) = -\varphi_1 r_1 + \varphi_2 r_2 - \varphi_3 r_3$, $J_2(\varphi) = \varphi_1 r_1 - \varphi_2 r_2 - \varphi_3 r_3$

For the localization of the parameters $\lambda = 4\pi^2 + \delta$, $A_1 = 1$, $A_2 = 4 + p$ by the bifurcation modes we have

$$e_1 = \sqrt{2}\,(\sin 2\pi s)\,r_1, \quad e_2 = \sqrt{2}\,(\sin \pi s)\,r_2$$

Here

$$\alpha_1(\delta, p) = -\delta, \quad \alpha_2(\delta, p) = -\delta + p$$

Condition (1.6) is not satisfied here. Elementary calculations (omitted here because of their length) show that $h^{\circ}_{1,2} < -\sqrt{h^{\circ}_{1,1}h^{\circ}_{2,2}}$. Therefore, the governing matrix in this example is not conditionally-positive in R_+^2.

The constraint $\mu > 4\pi^2$ formulated earlier "locks in" the mode

$$e_3(\mu) = d(\mu)(\cos\theta(\mu)(s - 1/_2) - \cos^1/_2\,\theta(\mu))\,r_1$$

Here $d(\mu)$ is a normalizing factor while $\theta(\mu)$ if found from the equation $\theta^2 = \mu(1 - 2\theta^{-1}\,\mathrm{tg}^1/_2\,\theta)$.

The situation of trimodal bifurcation with parallelepiped symmetry occurs for the localization $\mu \approx 4\pi^2$.

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